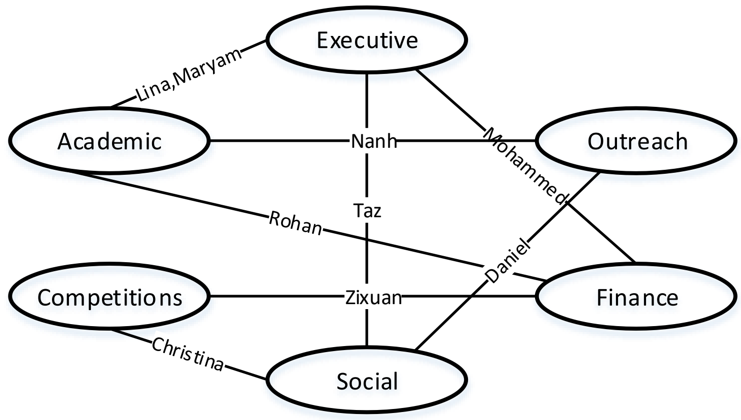
PART A – GRAPH THEORY

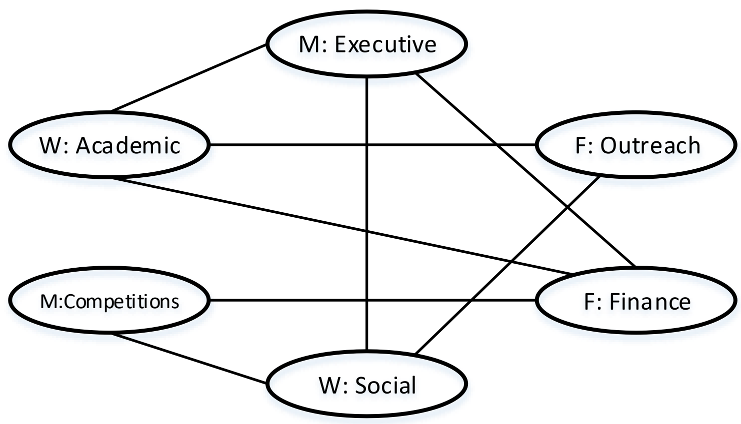
1. Committee Overlaps



Note that it was not necessary to label the edges to answer this question. They are labelled here to explain the answer.

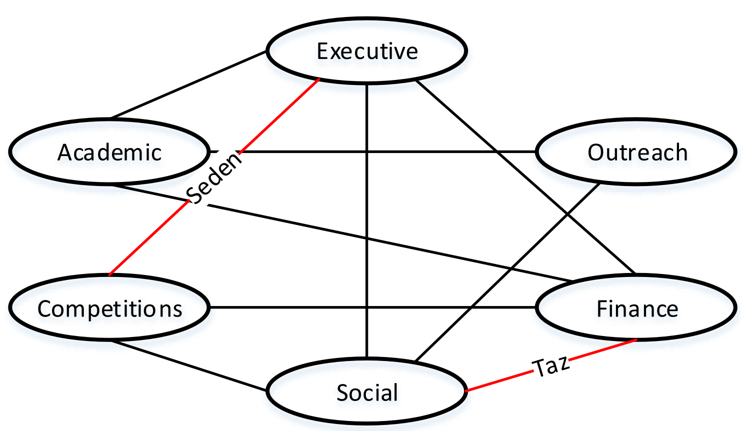
2. Committee Meetings

This question has more than one correct answer. One possible answer is shown in the graph below:



3. Modified Committee Overlaps

The two red lines should be added to the above graph (again edge labels are shown here to explain)



4. Cliques

a) 3-cliques: {E,A,F}, {E,S,F}, {E, C, F}, {E, C, S}, {C,F,S}
b) 4-cliques: {E, C, F, S}

5. Further Thoughts

Would it now be possible to schedule meetings for all of these committees in the three available time slots in such a way that all the student reps could attend all the meetings of the committees on which they are serving?

No because the graph contains a 4-clique. Adjacent vertices need to be scheduled on different days, and in the 4-clique each of the 4 vertices is adjacent to the other 3, so 4 different days are needed for the 4 committees in the 4-clique.

PART B – SEQUENCES, RECURRENCE RELATIONS

Given the sequence a_n defined with the recurrence relation:

$a_0 = 2$

$a_k = 4k + a_{k-1} + 2$ for $k \geq 1$

1. Terms of the Sequence

$a_1 = 4\times1 + a_0 + 2 = 4\times1 + 2 + 2 = 4\times1 + 2\times2 = 8$

$a_2 = 4\times2 + a_1 + 2 = 4\times2 + (4\times1 + 2\times2) + 2 = 4\times2 + 4\times1 + 3\times2 = 18$

$a_3 = 4\times3 + a_2 + 2 = 4\times3 + (4\times2 + 4\times1 + 3\times2) + 2 = 4\times3 + 4\times2 + 4\times1 + 4\times2 = 32$

$a_4 = 4\times4 + a_3 + 2 = 4\times4 + (4\times3 + 4\times2 + 4\times1 + 4\times2) + 2 = 4\times4 + 4\times3 + 4\times2 + 4\times1 + 5\times2 = 50$

2. Iteration

Using iteration, solve the recurrence relation when $n \geq 0$ (i.e. find an analytic formula for a_n). Simplify your answer as much as possible, showing your work. In particular, your final answer should not contain sums ($\sum$) and products ($\prod$)

$a_n = 4 \times \sum_{i=1}^{n} i + 2(n+1) = 4n(n+1)/2 + 2(n+1) = 2(n+1)^2$

PART C – INDUCTION

1. Set D

$D=\mathbb{N}^+$

2. P(n)

P(n) is: $4^n \bmod 10 = 4 \vee 4^n \bmod 10 = 6$

3. Basic Step of the Proof

When n=1 $4^n \bmod 10 = 4 \bmod 10 = 4$, so P(1) is true
When n=2 $4^n \bmod 10 = 16 \bmod 10 = 6$, so P(2) is true

4. Inductive Step of the Proof

Assume that some positive integer k is such that P(m) is true for all $m \leq k$
i.e. $\forall m \in \{1, \ldots, k\}\ 4^m \bmod 10 = 4 \vee 4^m \bmod 10 = 6$ (Inductive Hypothesis)

We will now show that P(k+1) is true, i.e. $4^{k+1} \bmod 10 = 4 \vee 4^{k+1} \bmod 10 = 6$
$4^{k+1} = 4 \times 4^k$

By Inductive Hypothesis P(k) is true, i.e. $4^k \bmod 10 = 4 \vee 4^k \bmod 10 = 6$

Case 1: $4^k \bmod 10 = 4$
this means $\exists a \in \mathbb{N}\ 4^k = 10a + 4$
so $4^{k+1} = 4 \times 4^k = 4(10a + 4) = 40a + 16 = 10(4a+1) + 6$
i.e. $4^{k+1} \bmod 10 = 6$

Case 2: $4^k \bmod 10 = 6$
this means $\exists a \in \mathbb{N}\ 4^k = 10a + 6$
so $4^{k+1} = 4 \times 4^k = 4(10a + 6) = 40a + 24 = 10(4a+2) + 4$
i.e. $4^{k+1} \bmod 10 = 4$

So $4^{k+1} \bmod 10 = 6 \vee 4^{k+1} \bmod 10 = 4$
i.e. P(k+1)
QED